Lecture 7

Recall For $G = SL_n(\mathbb{R})$ or $G = GL_n(\mathbb{R})$,
$\Gamma = SL_n(\mathbb{Z})$ or $\Gamma = GL_n(\mathbb{Z})$,

$P = MU$: standard parabolic subgroup of G | $P \leftrightarrow n = m_1 + \cdots + m_r$, $M \underset{G=GL_n}{\simeq} GL_{m_1} \times \cdots \times GL_{m_r}$

$\rightsquigarrow$ for each $\varphi : \Gamma \backslash G \to \mathbb{C}$,

$$\varphi_P(g) := \int_{\Gamma_U \backslash U} \varphi(ug)\, du \qquad \text{"constant term of } \varphi \text{ along } P\text{"}$$

Lemma 1 If φ is an automorphic form on $\Gamma \backslash G$, then φ_P is an "automorphic form on $\Gamma_P U \backslash G$":

(A$_1$) φ_P is left-invariant under U and Γ_P (NB $U \triangleleft P$, $\Gamma_P U = U \Gamma_P$)

(A$_2$) φ_P is right K-finite (with K-type determined by that of φ)

(A$_3$) $\forall\, g \in G$, the function
$$\Gamma_M \backslash M \longrightarrow \mathbb{C}$$
$$x \longmapsto \varphi_P(xg)$$
is $\mathfrak{Z}(\mathfrak{m})$-finite. ($\mathfrak{m} = \mathrm{Lie}(M)$) (with $\mathfrak{Z}(\mathfrak{m})$-type determined by $\mathfrak{Z}(\mathfrak{g})$-type of φ)

(A$_4$) moderate growth

Pf sketch (A$_1$): exercise w/ Haar measure

(A$_2$): ✓ if $\mathrm{Span}\{\varphi(\cdot k) : k \in K\} = V$, $\dim V < \infty$
then $\mathrm{span}\{\varphi_P(\cdot k) : k \in K\} = $ image of V under $f \mapsto f_P$.

because $\varphi_P(\cdot k) = (\varphi(\cdot k))_P$

(A$_4$): exercise

(A$_3$): Proof is a basic consequence of the main theorem describing $\mathfrak{Z}(\mathfrak{g})$, called the <u>Harish-Chandra isomorphism</u>.

(A$_3$) in simplest example: Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic modular form of wt m, $P = \begin{pmatrix} * & * \\ & * \end{pmatrix}$. $\quad f \leftrightarrow \varphi : \Gamma \backslash G \to \mathbb{C}$ aut. form

$$\varphi_P \iff a_0(iy) := \int_{x \in \mathbb{R}/\mathbb{Z}} f(x+iy)\, dx$$

Fact f: holomorphic $\overset{?}{\Longrightarrow}$ a_0: constant. Indeed, $a_0 : \mathbb{H} \to \mathbb{C}$, holomorphic invariant under translation by $\mathbb{R} \Rightarrow$ constant.

(apply Cauchy-Riemann equations in (x,y) coordinates)

Recall $\varphi : \Gamma\backslash G \to \mathbb{C}$ is *cuspidal* if $\varphi_P = 0 \;\forall\; P \subsetneq G$.

If $\varphi \in L^2$, then we interpret "$\varphi_P = 0$" as holding almost everywhere.

(NB if $P = G$, then $M = G$, $U = \{1\}$, so $\varphi_P = \varphi$.)

Lemma $\{\text{cuspidal } \varphi \in L^2(\Gamma\backslash G)\} =: L^2_{\text{cusp}}(\Gamma\backslash G)$ is a closed subspace.

Remark $\left\{\varphi \in L^2(\mathbb{R}^2/\mathbb{Z}^2) : \int_{x \in \mathbb{R}/\mathbb{Z}} \varphi(x,y)\,dx = 0 \text{ for a.e. } y\right\} =: V$

is closed in $L^2(\mathbb{R}^2/\mathbb{Z}^2)$. Explicitly,

$$V = \text{orthogonal complement of } \{\underbrace{(x,y) \mapsto f(y)}_{\Phi_f} : f \in C(\mathbb{R}/\mathbb{Z})\}.$$

$$\underset{\varphi}{\uparrow}$$

$$\int_{\mathbb{R}^2/\mathbb{Z}^2} \varphi\, \Phi_f = \int_{y \in \mathbb{R}/\mathbb{Z}} f(y) \left(\int_{x \in \mathbb{R}/\mathbb{Z}} \varphi(x,y)\,dx \right) dy$$

This Remark leads to a proof of the Lemma. We'll give another proof (related).

Defn $P = MU$: standard parabolic

$f : \Gamma_P U \backslash G \to \mathbb{C}$

$\rightsquigarrow \operatorname{Eis}(f) : \Gamma\backslash G \to \mathbb{C}$

$$g \longmapsto \sum_{\Gamma_P\backslash\Gamma} f(\gamma g), \text{ if abs. convergent.}$$

Exercise if $f \in C_c(\Gamma_P U\backslash G)$, then $\operatorname{Eis}(f) \in C_c(\Gamma\backslash G)$

Proof of Lemma reduces to the following assertion:

$$\varphi : \Gamma\backslash G \to \mathbb{C} \text{ is cuspidal} \iff \int_{\Gamma\backslash G} \varphi \operatorname{Eis}(f) = 0$$

$\forall$ standard $P \subsetneq G$, $f \in C_c(\Gamma_P U\backslash G)$.

reduces to the computation: $\displaystyle\int_{\Gamma\backslash G} \varphi \operatorname{Eis}(f) = \int_{\Gamma_P U\backslash G} \varphi_P\, f.$

$$\int_{\Gamma\backslash G} \varphi \, \mathrm{Eis}(f) \overset{?}{=} \int_{\Gamma_P U\backslash G} \varphi_P \, f$$

$$\| $$

$$\int_{g\in\Gamma\backslash G} \varphi(g) \sum_{\gamma\in\Gamma_P\backslash\Gamma} f(\gamma g)\, dg$$

$\|$ Fubini + left-Γ-invariance of φ

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma_P\backslash\Gamma} \varphi(\gamma g) f(\gamma g)\, dg$$

$$\|$$

$$\int_{\Gamma_P\backslash G} \varphi(g)\, f(g)\, \underbrace{dg}_{\text{right Haar measure}} \underset{\text{Fubini, } \Gamma_P\cap U = \Gamma_U}{=} \int_{g\in \Gamma_P U\backslash G} \int_{u\in\Gamma_U\backslash U} \varphi(ug) f(ug)\, \underbrace{du}_{\text{Haar measure}}\, dg$$

$\|$ $f(ug) = f(g)$

$$\int_{g\in\Gamma_P U\backslash G} f(g) \overbrace{\left(\int_{u\in\Gamma_U\backslash U} \varphi(ug)\, du\right)}^{\varphi_P(g)} dg \quad \checkmark$$

Finiteness theorems for automorphic forms

Generalizations of: $\dim\left(\left\{\begin{array}{c}\text{modular forms for } SL_2(\mathbb{Z}) \\ \text{of weight } m\end{array}\right\}\right) < \infty.$

Defn Let $\varphi : G \to \mathbb{C}$ be right K-finite.
Thus $V = \mathrm{span}\{\varphi(\cdot\, k) : k\in K\}$ is finite-dimensional space.
By Maschke's Theorem (using that K: compact), we have

$$V \cong \bigoplus_{j=1}^{\ell} V_j^{\oplus \mu_j}, \qquad V_j : \text{irreducible of } K, \; \underset{\text{right translation}}{\circlearrowleft} K$$

occurring w/ multiplicity $\mu_j \in \mathbb{N} = \{1,2,3,\dots\}$

Given a finite subset Ξ of the set of isom. classes of irreducible representations of K, we say that φ has K-type Ξ if each V_j lies in Ξ.

Another way to express this condition uses character theory for K.

Define $\Xi \rightsquigarrow \xi = \xi_{\Xi}$

$$\xi : K \longrightarrow \mathbb{C}$$
$$k \longmapsto \sum_{\sigma \in \Xi} \overbrace{(\dim \sigma)\, \overline{\chi_\sigma(k)}}^{e_\sigma},$$

$\chi_\sigma(k) = \operatorname{trace}(\sigma(k))$
(character of σ)

Then φ has K-type Ξ
$\Longleftrightarrow \varphi * \xi = \varphi.$

character theory: $\forall$ irreducible representations σ, τ of K, the operator

$$\tau(e_\sigma) : V_\tau \longrightarrow V_\tau$$
$$v \longmapsto \int_K e_\sigma(k)\, \tau(k) v \underbrace{dk}_{\text{prob. Haar}}$$

satisfies $\tau(e_\sigma) = \begin{cases} \text{identity} & \text{if } \sigma \cong \tau, \\ 0 & \text{else.} \end{cases}$

<u>Defn</u> Let $\varphi : G \to \mathbb{C}$ be $\mathfrak{Z}(\mathfrak{g})$-finite.
(f.g. $\mathbb{C}$-alg)

Set $J := \operatorname{Ann}_{\mathfrak{Z}(\mathfrak{g})}(\varphi)$. Then

$$\mathfrak{Z}(\mathfrak{g})/J \cong \underbrace{\mathfrak{Z}(\mathfrak{g}) \cdot \varphi}_{\text{f. dim'l}} = \{ D\varphi : D \in \mathfrak{Z}(\mathfrak{g}) \}.$$

$\Longrightarrow J$: finite codimension, ideal.

We say in general that φ has <u>$\mathfrak{Z}(\mathfrak{g})$-type J</u> if $J = \operatorname{Ann}_{\mathfrak{Z}(\mathfrak{g})}(\varphi)$.

Theorem (Harish-Chandra) $(\Gamma \backslash G = SL_n\mathbb{Z} \backslash SL_n\mathbb{R}$ OR $GL_n\mathbb{Z} \backslash GL_n\mathbb{R})$

Let Ξ, J: as above. Define

Then $\mathcal{A}(\Gamma\backslash G, \Xi, J) := \left\{ \begin{array}{l} \text{aut. forms } \varphi \text{ on } \Gamma\backslash G \text{ of K-type } \Xi \\ \mathfrak{Z}(\mathfrak{g})\text{-type } J \end{array} \right\}$

is a finite-dimensional vector space.

Proof sketch

- We can reduce from GL_n to SL_n using Lemma from last time. Case of SL_1 is trivial.
- Induct on n. More generally, we may assume inductively that $\forall P \subsetneq G$, "MU" $\forall \Xi$, $\forall J \subseteq \mathfrak{Z}(\mathfrak{m})$, M = product of GL_m's, $m < n$.

$$\dim \mathcal{A}(\Gamma_P U \backslash G, \Xi, J) < \infty.$$

- By Lemma 1, we deduce that $\forall P \subsetneq G$, $\forall \Xi$, $\forall J \subseteq \mathfrak{Z}(\mathfrak{g})$, the map

$$\mathcal{A}(\Gamma\backslash G, \Xi, J) \longrightarrow \{\text{aut. forms on } \Gamma_P U\backslash G\}$$

has finite-dimensional image. Thus

$$\mathcal{A}_0 := \mathcal{A}_0(\Gamma\backslash G, \Xi, J) = \bigcap_{P \subsetneq G} \ker(\text{above map})$$

has finite codimension. Reduce to checking that $\dim(\mathcal{A}_0) < \infty$.

- We've seen that every $\varphi \in \mathcal{A}_0$ lies in L^2 and is of rapid decay, in particular lies in L^∞. Quantifying that argument, we in fact have $\|\varphi\|_\infty \leq C \|\varphi\|_{L^2}$, $C = C(\Xi, J)$

- $\mathcal{A}_0$ is closed in $L^2(\Gamma\backslash G)$:
  - L^2_{cusp} is closed, as noted above
  - the condition "φ has K-type Ξ, $\mathfrak{Z}(\mathfrak{g})$-type J" is closed, because if $\varphi_n \in L^2_{cusp} \cap \mathcal{A}(\Xi, J)$ converges in L^2 to φ, then $D\varphi_n \xrightarrow[\infty]{} D\varphi$ $\forall D \in \mathcal{U}(\mathfrak{g})$.

· Final step:

<u>Lemma (Godement)</u> if $V \subseteq L^2(\Gamma \backslash G)$ is any closed subspace s.t. $\exists C: \|\varphi\|_\infty \leq C \|\varphi\|_2 \ \forall \varphi \in V$, then $\dim(V) < \infty$.